

# Comparative and qualitative robustness for law-invariant risk measures

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## Abstract

When estimating the risk of a P&L from historical data or Monte Carlo simulation, the robustness of the estimate is important. We argue here that Hampel's classical notion of qualitative robustness is not suitable for risk measurement and we propose and analyze a refined notion of robustness that applies to tail-dependent law-invariant convex risk measures on Orlicz space. This concept of robustness captures the tradeoff between robustness and sensitivity and can be quantified by an index of qualitative robustness. By means of this index, we can compare various risk measures, such as distortion risk measures, in regard to their degree of robustness. Our analysis also yields results that are of independent interest such as continuity properties and consistency of estimators for risk measures, or a Skorohod representation theorem for  $\psi$ -weak convergence.

**Key words:** Law-invariant risk measure, convex risk measure, coherent risk measure, Orlicz space, qualitative robustness, comparative robustness, index of qualitative robustness, Hampel's theorem,  $\psi$ -weak topology, distortion risk measure, Skorohod representation

## 1 Introduction

Let  $X$  denote the P&L of a financial position. When assessing the risk of  $X$  in terms of a monetary risk measure  $\rho$  it is common to estimate  $\rho(X)$  by means of a Monte Carlo procedure or from a sequence of historical data. When  $\rho$  is a law-invariant risk measure, a natural estimate for  $\rho(X)$  is given by  $\mathcal{R}_\rho(\hat{m})$ , where  $\hat{m}$  is the empirical distribution of the data and  $\mathcal{R}_\rho$  is the functional defined by

$$\mathcal{R}_\rho(\mu) = \rho(X) \quad \text{if } X \text{ has law } \mu;$$

see, e.g., [1, 2, 3, 9, 31]. In this context, it was pointed out by Cont et al. [9] that it is important to consider the *robustness* of the risk functional  $\mathcal{R}_\rho$ . Informally, robustness refers here to a certain insensitivity of the estimate with respect to deviations of  $\hat{m}$  from the 'true', theoretical distribution. It will especially yield a stable behavior of the estimates when the estimation process is repeated periodically. Such a stable behavior is particularly desirable when  $\rho(X)$  serves in allocating the

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economic capital required from a large position  $X$ , since altering the capital allocation may be costly.

On a mathematical level, Cont et al. [9] use Hampel's [19] classical concept of qualitative robustness, which, according to Hampel's theorem, is essentially equivalent to the weak continuity of  $\mathcal{R}_\rho$ . Consequently, it was pointed out in [9] that no risk functional  $\mathcal{R}_\rho$  that arises from a law-invariant coherent risk measure  $\rho$  can satisfy Hampel's requirement of qualitative robustness, not even if  $\rho$  is simply the ordinary expectation of the loss. The results in [9] therefore seem to weigh heavily in favor of Value at Risk, since Value at Risk does essentially satisfy Hampel's notion of robustness.

Our goal in this article is to point out that in risk measurement the use of Hampel's classical concept of qualitative robustness may be problematic and to propose and analyze an alternative concept based on [22]. Let us start by discussing two major drawbacks of Hampel's robustness in risk measurement.

First, two P&Ls may have laws that are close with respect to the weak topology but still have completely different tail behavior. Qualitative robustness of  $\mathcal{R}_\rho$  therefore requires that  $\rho$  is essentially insensitive to the tail behavior of a P&L. In the recent years of financial crisis, it has become apparent, though, that an faulty assessment of tail behavior can lead to a dramatic underestimation of the corresponding risk.

Second, Hampel's robustness concept creates a sharp division of the class of law-invariant monetary risk measures into those for which  $\mathcal{R}_\rho$  is 'robust' and those for which  $\mathcal{R}_\rho$  is 'not robust'. The first class contains risk measures such as Value at Risk that are insensitive with respect to tail behavior of P&Ls whereas the second class contains the ordinary expectation and all law-invariant coherent risk measures [9]. But, as we will see, the distinction between 'robust' and 'non-robust' risk measures is artificial because there is actually a full continuum of possible degrees of robustness beyond the classical concept. So labeling a risk measure as 'robust' or 'non-robust' may give a false impression.

In this article, we will analyze the robustness properties of law-invariant convex risk measures based on the refined notion of qualitative robustness that was proposed in [22] and is further developed here in Section 2.4. Instead of a sharp division into 'robust' and 'non-robust' risk measures, this notion allows us to assign a degree of robustness to most risk measures and to compare different risk measures in regard to their degrees of robustness. We thereby capture the natural tradeoff between robustness and tail sensitivity in risk measurement. The degree of robustness can be expressed numerically by the *index of qualitative robustness* proposed in [22]. This index takes values in  $[0, \infty]$ , with the respective extremes  $+\infty$  and  $0$  corresponding to Hampel's robustness and to full tail sensitivity. Some of our main results will show that a greater index of qualitative robustness implies greater robustness in a sense that is mathematically precise. We will also show how our index can be computed for distortion risk measures such as MINMAXVAR or Average Value at Risk (which is also called Expected Shortfall, Conditional Value at Risk, or TailVaR).

Since we are interested in the tail dependence of convex risk measures, it is not sufficient to consider only bounded P&Ls. We therefore build on the analysis of Cheridito and Li [6], who observed that Orlicz spaces or Orlicz hearts are appropriate domains for convex risk measures when P&Ls are unbounded. We are particularly interested in the continuity properties of  $\rho$  and its corresponding risk functional  $\mathcal{R}_\rho$ , and we find that an important role is played by the so-called  $\Delta_2$ -condition of the underlying Orlicz space.

Our analysis also yields some results that are of independent interest. For instance, we obtain

a Skorohod representation theorem that links  $\psi$ -weak convergence of probability measures to norm convergence of random variables in Orlicz space.

Our article is organized as follows. In Section 2.2 we prove the consistency of the estimator  $\mathcal{R}_\rho(\widehat{m})$  for law-invariant convex risk measures and general stationary and ergodic data. In Section 2.3 we analyze the continuity properties of  $\mathcal{R}_\rho$  and show that  $\mathcal{R}_\rho$  basically inherits the continuity of the original risk measures if and only if the Orlicz space supporting  $\rho$  satisfies the  $\Delta_2$ -condition. In Section 2.4 we present our main results on the comparative and qualitative robustness of law-invariant convex risk measures  $\rho$ . In particular, we introduce our refined notion of robustness and the index of qualitative robustness, and we show that these notions are well-defined whenever the Orlicz space supporting  $\rho$  satisfies the  $\Delta_2$ -condition. In Section 2.5 we show that our results can be easily applied to distortion risk measures.

In Section 3.1 we continue and strengthen the robustness analysis for general statistical functionals started in [22]. In particular, we state stronger versions of our Hampel-type theorem and its converse than those given in [22]. In Section 3.2 we state and prove our above-mentioned Skorohod representation result. Most other proofs can be found in Section 4 and the appendix.

## 2 Statement of main results

### 2.1 Setup

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be an atomless probability space and denote by  $L^0 := L^0(\Omega, \mathcal{F}, \mathbb{P})$  the usual class of all finitely-valued random variable modulo the equivalence relation of  $\mathbb{P}$ -a.s. identity. Let  $\mathcal{X} \subset L^0$  be a vector space containing the constants. An element  $X$  of  $\mathcal{X}$  will be interpreted as the P&L of a financial position. We will say that a map  $\rho : \mathcal{X} \rightarrow \mathbb{R}$  is a *convex risk measure* when the following conditions are satisfied:

- (i) monotonicity:  $\rho(X) \geq \rho(Y)$  for  $X, Y \in \mathcal{X}$  with  $X \leq Y$ ;
- (ii) convexity:  $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$  for all  $X, Y \in \mathcal{X}$  and  $\lambda \in [0, 1]$ ;
- (iii) cash coercivity:  $\rho(-m) \rightarrow +\infty$  when  $m \in \mathbb{R}$  tends to  $+\infty$ .

In the standard definition of a convex risk measure, cash coercivity is replaced by the stronger requirement of

- (iii') cash additivity:  $\rho(X + m) = \rho(X) - m$  for  $X \in \mathcal{X}$  and  $m \in \mathbb{R}$ .

But it was argued in [15] that the requirement of cash additivity should be relaxed when interest rates are stochastic or ambiguous, or when bonds are subject to possible default. We will therefore not insist on cash additivity and work with the weaker axiom (iii) instead. Our examples, however, will be formulated within the framework of cash-additive convex or coherent risk measures.

As discussed, e.g., in [1, 2, 3, 9] it is a common procedure to estimate the risk of a financial position by means of a Monte Carlo procedure or from a sequence of historical data. Such a procedure makes sense when  $\rho$  is *law-invariant*:  $\rho(X) = \rho(\widetilde{X})$  whenever  $X$  and  $\widetilde{X}$  have the same law under  $\mathbb{P}$ . Let us denote by  $\mathcal{M}(\mathcal{X}) := \{\mathbb{P} \circ X^{-1} : X \in \mathcal{X}\}$  the class of all Borel probability measures on  $\mathbb{R}$  that arise as the distribution of some  $X \in \mathcal{X}$ . Law invariance of a risk measure  $\rho : \mathcal{X} \rightarrow \mathbb{R}$  is equivalent to the existence of a map  $\mathcal{R}_\rho : \mathcal{M}(\mathcal{X}) \rightarrow \mathbb{R}$  such that

$$(2.1) \quad \rho(X) = \mathcal{R}_\rho(\mathbb{P} \circ X^{-1}), \quad X \in \mathcal{X}.$$

This map  $\mathcal{R}_\rho$  will be called the *risk functional* associated with  $\rho$ . It can be used in a natural way to construct estimates for the risk  $\rho(X)$  of  $X \in \mathcal{X}$ . All one has to do is to take an estimate  $\hat{\mu}_n$  for the law  $\mu = \mathbb{P} \circ X^{-1}$  of  $X$  and to plug this estimate into the functional  $\mathcal{R}_\rho$  to get the desired estimator:

$$(2.2) \quad \hat{\rho}_n := \mathcal{R}_\rho(\hat{\mu}_n);$$

see, e.g., [1, 2, 3, 9, 31]. For instance,  $\hat{\mu}_n$  can be the empirical distribution  $\frac{1}{n} \sum_{k=1}^n \delta_{x_k}$  of a sequence  $x_1, \dots, x_n$  of historical observations or Monte Carlo simulations.

**Example 2.1.** In this very basic example we take  $\mathcal{X} = L^1$  and  $\rho(X) = -\mathbb{E}[X]$ . Then  $\rho$  is a law-invariant coherent risk measure and  $\mathcal{R}_\rho(\mu)$  is simply the negative mean of the measure  $\mu$ , i.e.,  $\mathcal{R}_\rho(\mu) = -\int x \mu(dx)$ .

There are two natural questions that arise in this context. The first question refers to the *consistency* of a sequence of estimates  $\hat{\rho}_n$ . That is, under which conditions do we have  $\hat{\rho}_n \rightarrow \rho(X)$  as  $n \uparrow \infty$ ? When we assume that the estimates  $\hat{\mu}_n$  converge to  $\mu$  in some suitable topology on the space of measures, then the consistency of  $\hat{\rho}_n$  boils down to establishing the continuity of  $\mathcal{R}_\rho$  in that topology. We will thus also analyze the continuity properties of  $\mathcal{R}_\rho$ .

Once consistency and continuity have been established, one can investigate the *robustness* of the estimate  $\hat{\rho}_n$ . Informally robustness refers to the stability of  $\hat{\rho}_n$  with respect to small perturbations of the law under which the data points  $x_1, \dots, x_n$  are generated. The issue of robustness of this plug-in method for risk measures was first raised in [9]. Here we will address it in Section 2.4.

Before stating our results, we need to specify the setting in which we are going to work. A common choice for  $\mathcal{X}$  is the space  $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$  of all bounded random variables. When dealing with possibly unbounded risks, however, the choice  $\mathcal{X} = L^\infty$  is not suitable. It was observed in [4, 6] that Orlicz spaces or Orlicz hearts may be appropriate choices for  $\mathcal{X}$  when risks may be unbounded. Let us thus recall the basic notions of Orlicz spaces. Following [6], a Young function will be a left-continuous, nondecreasing convex function  $\Psi : \mathbb{R}_+ \rightarrow [0, \infty]$  such that  $0 = \Psi(0) = \lim_{x \downarrow 0} \Psi(x)$  and  $\lim_{x \uparrow \infty} \Psi(x) = \infty$ . Such a function is continuous except possibly at a single point at which it jumps to  $+\infty$ . The Orlicz space associated with  $\Psi$  is

$$L^\Psi := L^\Psi(\Omega, \mathcal{F}, \mathbb{P}) = \{X \in L^0 : \mathbb{E}[\Psi(c|X|)] < \infty \text{ for some } c > 0\}.$$

It is a Banach space when endowed with the Luxemburg norm,

$$\|Y\|_\Psi := \inf \{\lambda > 0 : \mathbb{E}[\Psi(|Y|/\lambda)] \leq 1\}.$$

The Orlicz heart is defined as

$$H^\Psi := H^\Psi(\Omega, \mathcal{F}, \mathbb{P}) = \{X \in L^0 : \mathbb{E}[\Psi(c|X|)] < \infty \text{ for all } c > 0\}.$$

When  $\Psi$  takes the value  $+\infty$ , then  $H^\Psi = \{0\}$  and  $L^\Psi = L^\infty$ . For this reason, we will mainly focus on the case in which  $\Psi$  is finite. Then  $L^\infty \subset H^\Psi \subset L^\Psi \subset L^1$ , and these inclusions may all be strict. In fact, the identity  $H^\Psi = L^\Psi$  holds if and only if  $\Psi$  satisfies the so-called  $\Delta_2$ -condition,

$$(2.3) \quad \text{there are } C, x_0 > 0 \text{ such that } \Psi(2x) \leq C\Psi(x) \text{ for all } x \geq x_0;$$

see [14, Theorem 2.1.17 (b)]. This condition is clearly satisfied when specifically  $\Psi(x) = x^p/p$  for some  $p \in [1, \infty)$ . In this case,  $H^\Psi = L^\Psi = L^p$  and  $\|Y\|_\Psi = p^{-1/p} \|Y\|_p$ .

**Example 2.2** (Risk measure based on one-sided moments). The risk measure based on one-sided moments is defined as

$$(2.4) \quad \rho(X) := -\mathbb{E}[X] + a \mathbb{E}[(X - \mathbb{E}[X])^-]^p]^{1/p},$$

where  $p \in [1, \infty)$  and  $a \in [0, 1]$  are constants; see also [12]. It is well-defined and finite on  $L^p$ , law-invariant, and it is easily seen that it satisfies the axioms of a convex risk measure.  $\diamond$

When the  $\Delta_2$ -condition (2.3) is not satisfied, then the Orlicz heart rather than the Orlicz space  $L^\Psi$  is the natural domain for a convex risk measure as is illustrated by the following examples.

**Example 2.3** (Entropic risk measure). The entropic risk measure is defined as

$$(2.5) \quad \rho(X) := \frac{1}{\beta} \log \mathbb{E}[e^{-\beta X}],$$

where  $\beta$  is a positive constant; see [17, Example 12]. It is well-defined and finite on the Orlicz heart  $H^\Psi$  for the Young function  $\Psi(x) = e^x - 1$ , but it is clearly not finite on the entire Orlicz space  $L^\Psi$ . The associated risk functional  $\mathcal{R}_\rho : \mathcal{M}(H^\Psi) \rightarrow \mathbb{R}$  is given by  $\mathcal{R}_\rho(\mu) = \frac{1}{\beta} \log \int e^{-\beta x} \mu(dx)$ .  $\diamond$

**Example 2.4** (Utility-based shortfall risk). The utility-based shortfall risk measure with loss function  $\ell$  was introduced in [17] as

$$(2.6) \quad \rho(X) := \inf\{m \in \mathbb{R} : \mathbb{E}[\ell(-X - m)] \leq x_0\}$$

for  $X \in L^\infty$ , where  $\ell : \mathbb{R} \rightarrow \mathbb{R}$  is nondecreasing, not identically constant, and  $x_0$  belongs to the interior of  $\ell(\mathbb{R})$ ; see also Section 4.9 in [18]. By taking  $\ell(x) = e^{\beta x}$  we recover the entropic risk measure (2.5). In the general case, we can define a finite Young function  $\Psi(x) := \ell(x) - \ell(0)$  for  $x \geq 0$ . With this choice,  $\rho(X)$  is well-defined and finite for each  $X \in H^\Psi$ . Indeed, we have

$$(2.7) \quad 0 \leq \ell(-X - m) \leq \frac{1}{2}\ell(-2X) + \frac{1}{2}\ell(-2m) \leq \frac{1}{2}\Psi(2|X|) + \frac{1}{2}\Psi(2|m|) + \ell(0),$$

which implies that  $\mathbb{E}[\ell(-X - m)]$  is finite for  $m \in \mathbb{R}$  and  $X \in H^\Psi$ . It is now easy to see that  $\rho$  is in fact a convex risk measure on  $H^\Psi$ . But when  $\Psi$  does not satisfy the  $\Delta_2$ -condition (2.3), then  $\rho(X)$  need not be finite for each  $X \in L^\Psi$ .  $\diamond$

## 2.2 Consistency

Let  $\rho$  be a law-invariant convex risk measure on  $H^\Psi$ , where  $\Psi$  is a finite Young function. We start by discussing the consistency of estimating the risk  $\rho(X)$  from a stationary and ergodic sequence  $X_1, X_2, \dots$  in  $H^\Psi$  (see [5, Section 6.7] for the definition of a stationary and ergodic process). This is a natural question if one wishes to estimate  $\rho(X)$  from historical data or from Monte Carlo simulations, where  $X$  is a random variable with the same law as  $X_i$ . Recall that every i.i.d. sequence is stationary and ergodic, and that ergodicity is implied by various mixing conditions.

We denote by

$$(2.8) \quad \widehat{m}_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

the empirical distribution of  $X_1, \dots, X_n$  and by

$$(2.9) \quad \widehat{\rho}_n := \mathcal{R}_\rho(\widehat{m}_n)$$

the corresponding estimate for  $\rho(X)$ .

**Theorem 2.5.** *Suppose that  $\rho$  is a law-invariant convex risk measure on  $H^\Psi$  and  $X_1, X_2, \dots$  is a stationary and ergodic sequence of random variables with the same law as  $X \in H^\Psi$ . Then (2.9) is a consistent estimator for  $\rho(X)$  in the sense that  $\widehat{\rho}_n \rightarrow \rho(X)$   $\mathbb{P}$ -a.s.*

In Theorem 2.5 it is essential that the sequence  $X_1, X_2, \dots$  satisfies a strong law of large numbers such as Birkhoff's ergodic theorem or Kolmogorov's law of large numbers. Perhaps surprisingly, it will in general not suffice to take just any reasonable estimating sequence  $(\widehat{\mu}_n)$  for  $\mathbb{P} \circ X^{-1}$  to obtain the consistency  $\mathcal{R}_\rho(\widehat{\mu}_n) \rightarrow \rho(X)$ . This is due to the possible failure of continuity of the map  $\mu \mapsto \mathcal{R}_\rho(\mu)$  when  $\Psi$  is not chosen suitably. We will give a precise meaning to this in our Theorem 2.7, where we analyze the continuity properties of the map  $\mu \mapsto \mathcal{R}_\rho(\mu)$ . These continuity properties will also be crucial for our subsequent discussion of the robustness of the estimators (2.9).

### 2.3 Continuity properties of $\mathcal{R}_\rho$

The basic issue when discussing the continuity of  $\mathcal{R}_\rho$  can already be observed in Example 2.1. There the map  $\mathcal{R}_\rho(\mu) = -\int x \mu(dx)$  is not continuous with respect to the standard weak topology of measures. We therefore need to use a stronger topology, a fact that was already observed in [31]. More precisely, we will consider the  $\psi$ -weak topology associated with a *weight function*  $\psi$ , i.e., a continuous function  $\psi : \mathbb{R} \rightarrow [0, \infty)$  satisfying  $\psi \geq 1$  outside some compact set. We denote by  $\mathcal{M}_1^\psi := \mathcal{M}_1^\psi(\mathbb{R})$  the class of all probability measures  $\mu$  on  $\mathbb{R}$  for which  $\int \psi d\mu < \infty$ . It coincides with the set  $\mathcal{M}_1 := \mathcal{M}_1(\mathbb{R})$  of all probability measures on  $\mathbb{R}$  if and only if  $\psi$  is bounded.

Furthermore,  $C_\psi(\mathbb{R})$  will denote the space of all continuous functions  $f$  on  $\mathbb{R}$  for which  $\sup_{x \in \mathbb{R}} |f(x)/(1 + \psi(x))| < \infty$ . The  $\psi$ -weak topology on  $\mathcal{M}_1^\psi$  is the coarsest topology for which all mappings  $\mu \mapsto \int f d\mu$  with  $f \in C_\psi(\mathbb{R})$  are continuous; cf. Section A.6 in [18]. Clearly, the  $\psi$ -weak topology is finer than the weak topology, and the two topologies coincide if and only if  $\psi$  is bounded; see Appendix A.2 for details. When  $\Psi$  is a Young function, then  $\Psi(|\cdot|)$  is a weight function, and we will simply write  $\mathcal{M}_1^\Psi$  in place of  $\mathcal{M}_1^{\Psi(|\cdot|)}$ . We will also use the term  $\Psi$ -weak convergence instead of  $\Psi(|\cdot|)$ -weak convergence etc. We recall the notation

$$\mathcal{M}(H^\Psi) = \{\mathbb{P} \circ X^{-1} : X \in H^\Psi\}$$

for the class of all laws of random variables  $X \in H^\Psi$ .

**Remark 2.6.** For any Young function  $\Psi$ , the identity  $\mathcal{M}(H^\Psi) = \mathcal{M}_1^\Psi$  holds if and only if  $\Psi$  satisfies the  $\Delta_2$ -condition (2.3). Indeed, since the underlying probability space is atomless,  $\mathcal{M}_1^\Psi$  coincides with the set of the laws of all random variables  $X$  with  $\mathbb{E}[\Psi(|X|)] < \infty$ . But by [14, Theorem 2.1.17] this class of random variables coincides with  $H^\Psi$  if and only if the  $\Delta_2$ -condition holds.  $\diamond$



**Theorem 2.7.** *For a finite Young function  $\Psi$  the following conditions are equivalent.*

- (a) *For every law-invariant convex risk measure  $\rho$  on  $H^\Psi$ , the map  $\mathcal{R}_\rho : \mathcal{M}(H^\Psi) \rightarrow \mathbb{R}$  is continuous for the  $\Psi$ -weak topology.*
- (b)  *$\Psi$  satisfies the  $\Delta_2$ -condition (2.3).*

**Remark 2.8.** Fix  $p \in [1, \infty)$  and let  $\Psi_p(x) = x^p/p$ . According to [29, Theorem 7.12], the  $\Psi_p$ -weak topology is generated by the *Wasserstein metric* of order  $p$ ,

$$d_{W_p}(\mu, \nu) := \inf \left\{ \left( \int |x - y|^p \pi(dx, dy) \right)^{1/p} : \pi \in \mathcal{M}_1(\mathbb{R} \times \mathbb{R}) \text{ with marginals } \mu, \nu \right\}.$$

Since  $\Psi_p$  satisfies the  $\Delta_2$ -condition (2.3), Theorem 2.7 implies that  $\mathcal{R}_\rho$  is continuous with respect to  $d_{W_p}$  whenever  $\rho$  is a law-invariant convex risk measure on  $L^p$ . A corresponding result for  $p = \infty$  is stated in [31, Lemma 2.4].  $\diamond$

In Theorem 2.7, a risk measure  $\rho$  is given on some Orlicz heart  $H^\Psi$ , and it is shown that  $\mathcal{R}_\rho$  is continuous with respect to the  $\Psi$ -weak topology. But one could ask whether  $\mathcal{R}_\rho$  is even continuous with respect to a weaker topology. For instance, this would be the case when  $\rho$  can be extended to a law-invariant convex risk measure on a larger Orlicz heart  $H^\Phi \supset H^\Psi$ .

To address this question, we consider the generic situation in which  $\rho$  is a law-invariant convex risk measure on  $L^\infty$  and let

$$(2.10) \quad \bar{\rho} : L^1 \longrightarrow \mathbb{R} \cup \{+\infty\}$$

denote the unique extension of  $\rho$  that is convex, monotone, and lower semicontinuous with respect to the  $L^1$ -norm. The existence of such an extension was proved in [16]. When  $\bar{\rho}$  is finite on some Orlicz heart  $H^\Psi$ , it will be continuous on  $H^\Psi$  with respect to the corresponding Luxemburg norm by [6, Theorem 4.1], and so it will also be cash additive on  $H^\Psi$  when  $\rho$  is cash additive on  $L^\infty$ .

**Theorem 2.9.** *Suppose that  $\rho$  is a law-invariant convex risk measure on  $L^\infty$ . Let furthermore  $\Psi$  be a Young function satisfying the  $\Delta_2$ -condition (2.3). Then the following conditions are equivalent.*

- (a)  *$\bar{\rho}$  is finite on  $H^\Psi$ .*
- (b) *The map  $\mathcal{R}_{\bar{\rho}} : \mathcal{M}(H^\Psi) \rightarrow \mathbb{R}$  is continuous for the  $\Psi$ -weak topology.*
- (c) *The map  $\mathcal{R}_\rho : \mathcal{M}(L^\infty) \rightarrow \mathbb{R}$  is continuous for the  $\Psi$ -weak topology.*
- (d) *If  $(X_n)$  is a sequence in  $L^\infty$  with  $\|X_n\|_\Psi \rightarrow 0$ , then  $\rho(X_n) \rightarrow \rho(0)$ .*

## 2.4 Qualitative and comparative robustness

Informally, Hampel's classical concept of qualitative robustness of an estimator requires that a small change in the law of the data results in only small changes in the law of the estimator. For a precise statement, it will be convenient to assume that the data arises from an i.i.d. sequence of random variables  $(X_i)$ . We can then assume without loss of generality that the underlying probability space has a product structure:  $\Omega = \mathbb{R}^{\mathbb{N}}$ ,  $X_i(\omega) = \omega(i)$  for  $\omega \in \Omega$  and  $i \in \mathbb{N}$ , and  $\mathcal{F} := \sigma(X_1, X_2, \dots)$ . For any Borel probability measure  $\mu$  on  $\mathbb{R}$ , we will denote by

$$\mathbb{P}_\mu := \mu^{\otimes \mathbb{N}}$$

the corresponding product measure under which the  $X_i$  are i.i.d. with common distribution  $\mu$ . Then  $(\Omega, \mathcal{F}, \mathbb{P}_\mu)$  is atomless unless  $\mu$  is a Dirac measure. We will retain the definitions (2.8) and (2.9) for the empirical distribution  $\hat{m}_n$  and for the corresponding risk estimate  $\hat{\rho}_n$  in this setting. We can now state the following version of Hampel's definition of qualitative robustness, suitably adapted to our more general setting.

**Definition 2.10** (Qualitative robustness). Let  $\mathcal{N} \subset \mathcal{M}_1$  be a set of probability measures. Let furthermore  $d_A$  be a metric on  $\mathcal{N}$  and  $d_B$  be a metric on  $\mathcal{M}_1$ . Then the risk functional  $\mathcal{R}_\rho$  is called *robust on  $\mathcal{N}$  with respect to  $d_A$  and  $d_B$*  if for all  $\mu \in \mathcal{N}$  and  $\varepsilon > 0$  there exists  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that

$$(2.11) \quad \nu \in \mathcal{N}, \quad d_A(\mu, \nu) \leq \delta \quad \implies \quad d_B(\mathbb{P}_\mu \circ \hat{\rho}_n^{-1}, \mathbb{P}_\nu \circ \hat{\rho}_n^{-1}) \leq \varepsilon \quad \text{for } n \geq n_0.$$

In the classical literature on qualitative robustness [19, 20, 11, 24] and in [9], the distances  $d_A$  and  $d_B$  are chosen so that they generate the weak topology of measures. For instance, they are taken as the Prohorov or Lévy metrics, and  $\mathcal{N}$  is taken as  $\mathcal{M}_1$ . It is a consequence of Hampel's celebrated theorem that such a choice essentially limits the concept of robustness to risk functionals that are continuous for the weak topology; see, e.g., [20, Theorem 2.21].

Insisting on metrics for the weak topology when assessing the robustness of a risk functional  $\mathcal{R}_\rho$  is problematic for a number of reasons.

First, two distributions  $\mu$  and  $\nu$  can be rather close with respect to a distance  $d_A$  for the weak topology, but still have completely different tail behavior. In this case, robustness (2.11) requires that  $\mathcal{R}_\rho(\mu)$  is *insensitive to the tail behavior* of  $\mu$ . This can be seen immediately, e.g., from the definition of the Lévy metric,

$$d_{\text{Lévy}}(\mu, \nu) = \inf \{ \varepsilon > 0 : F_\mu(x - \varepsilon) - \varepsilon \leq F_\nu(x) \leq F_\mu(x + \varepsilon) + \varepsilon \text{ for all } x \},$$

where  $F_\mu$  and  $F_\nu$  are the distribution functions for  $\mu$  and  $\nu$ . In the recent years of financial crisis, it has become apparent, though, that a misspecification of the tail behavior of a P&L can lead to a dramatic underestimation of the associated risk.

Second, linking the terminology of robustness exclusively to the weak topology generates a sharp but artificial division of risk functionals into the class of those that are “robust” and another class of those that are “not robust”. The first class contains risk functionals such as Value at Risk which are insensitive to the tail behavior of P&Ls, whereas the second class contains, e.g., the ordinary expectation and all law-invariant coherent risk measures [9]. This classification thus seems to provide a strong argument in favor of Value at Risk and against coherent risk measures.



We will see, however, that the distinction between “robust” and “non-robust” risk functionals is artificial because there is actually a full continuum of possible degrees of robustness beyond the classical concept. This new look at robustness will then help us to bring the argument against coherent risk measures back into perspective: robustness is not lost entirely but only to some degree when Value at Risk is replaced by a coherent risk measure such as Average Value at Risk.

It was observed in [22] that the basic idea for addressing this problem is to choose suitable metrics in (2.11). For  $d_B$  we will take the Prohorov distance, which is defined as

$$d_{\text{Proh}}(\mu, \nu) := \inf \{ \varepsilon > 0 : \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(\mathbb{R}) \}.$$

Note that  $d_{\text{Proh}} \geq d_{\text{Lévy}}$ , so our choice  $d_B := d_{\text{Proh}}$  gives a stronger notion of robustness than the choice  $d_B := d_{\text{Lévy}}$ , which would just as well be possible. For  $d_A$ , we will choose the *Prohorov  $\psi$ -metric*,

$$(2.12) \quad d_\psi(\mu, \nu) := d_{\text{Proh}}(\mu, \nu) + \left| \int \psi d\mu - \int \psi d\nu \right|, \quad \mu, \nu \in \mathcal{M}_1^\psi,$$

which by Lemma A.3 metrizes the  $\psi$ -weak topology on  $\mathcal{M}_1^\psi$  for a given weight function  $\psi$ . Also in (2.12) we could have replaced  $d_{\text{Proh}}$  by  $d_{\text{Lévy}}$ , but here the advantage of  $d_{\text{Proh}}$  is that it extends to a multivariate setting; see [22]. When  $\psi(x) = \Psi(|x|)$  for some Young function  $\Psi$ , we will simply write  $d_\Psi$  instead of  $d_{\Psi(|\cdot|)}$ . We will also need the following notion.

**Definition 2.11.** Let  $\psi$  be a weight function. A set  $\mathcal{N} \subset \mathcal{M}_1^\psi$  is called *uniformly  $\psi$ -integrating* when

$$(2.13) \quad \lim_{M \rightarrow \infty} \sup_{\nu \in \mathcal{N}} \int_{\{\psi \geq M\}} \psi d\nu = 0.$$

When  $\psi$  is bounded, every set  $\mathcal{N} \subset \mathcal{M}_1^\psi = \mathcal{M}_1$  is uniformly  $\psi$ -integrating. But when  $\psi$  is not bounded, then the uniform continuity in (2.11) will typically hold only on uniformly  $\psi$ -integrating sets  $\mathcal{N}$ . Let us therefore introduce the following concept of robustness.

**Definition 2.12.** Let  $\psi$  be a weight function and  $\mathcal{M} \subset \mathcal{M}_1^\psi$ . A risk functional  $\mathcal{R}_\rho$  is called  *$\psi$ -robust on  $\mathcal{M}$*  when  $\mathcal{R}_\rho$  is robust with respect to  $d_\psi$  and  $d_{\text{Proh}}$  on every uniformly  $\psi$ -integrating set  $\mathcal{N} \subset \mathcal{M}$ .

We can now state the following preliminary result.

**Proposition 2.13.** Let  $\mathcal{R}_\rho$  be the risk functional associated with a law-invariant convex risk measure  $\rho$  on  $L^\infty$ . When  $\psi : \mathbb{R}_+ \rightarrow (0, \infty)$  is a nondecreasing function such that  $\mathcal{R}_\rho$  is  $\psi(|\cdot|)$ -robust on  $\mathcal{M}(L^\infty)$ , then  $\psi$  has at least linear growth:  $\liminf_{x \uparrow \infty} \psi(x)/x > 0$ .

The significance of the preceding proposition is that it allows us to essentially limit the analysis of the  $\psi$ -robustness of risk functionals to weight functions  $\psi(x) = \Psi(|x|)$  arising from a Young function  $\Psi$ . In this context, we have the following result.

**Theorem 2.14.** For a finite Young function  $\Psi$ , the following conditions are equivalent.

- (a) For every law-invariant convex risk measure  $\rho$  on  $H^\Psi$ ,  $\mathcal{R}_\rho$  is  $\Psi$ -robust on  $\mathcal{M}(H^\Psi)$ .
- (b)  $\Psi$  satisfies the  $\Delta_2$ -condition (2.3).

As in Theorem 2.9,  $H^\Psi$  may not be the “canonical” space for  $\rho$  in the sense that  $\rho$  can be extended to a larger space. Such a situation has an impact on the robustness of  $\rho$  as explained in the next result. By  $\bar{\rho}$  we denote again the extension (2.10).

**Theorem 2.15.** *Let  $\Psi$  be a Young function satisfying the  $\Delta_2$ -condition (2.3). For a law-invariant convex risk measure  $\rho$  on  $L^\infty$ , the following conditions are equivalent.*

- (a)  $\mathcal{R}_{\bar{\rho}}$  is  $\Psi$ -robust on  $\mathcal{M}_1^\Psi$ .
- (b)  $\mathcal{R}_\rho$  is  $\Psi$ -robust on  $\mathcal{M}(L^\infty)$ .
- (c)  $\bar{\rho}$  is finite on  $H^\Psi$ .

The most important aspect of Theorem 2.15 is that it allows us to study the robustness properties of a given risk functional on  $\mathcal{M}(L^\infty)$  rather than on its full domain. Since any risk functional that arises from a law-invariant convex risk measure is defined on  $\mathcal{M}(L^\infty)$ , we can thus compare two risk functionals in regard to their degree of robustness.

**Definition 2.16** (Comparative robustness). Suppose that  $\rho_1$  and  $\rho_2$  are two law-invariant convex risk measures on  $L^\infty$ . We will say that  $\rho_1$  is *at least as robust as*  $\rho_2$  if the following implication holds. When  $\Psi$  is a Young function satisfying the  $\Delta_2$ -condition (2.3), and  $\mathcal{R}_{\rho_2}$  is  $\Psi$ -robust on  $\mathcal{M}(L^\infty)$ , then  $\mathcal{R}_{\rho_1}$  is  $\Psi$ -robust on  $\mathcal{M}(L^\infty)$ . When, in addition, there is a  $\Psi$  such that  $\mathcal{R}_{\rho_1}$  is  $\Psi$ -robust on  $\mathcal{M}(L^\infty)$  but  $\mathcal{R}_{\rho_2}$  is not, then we will say that  $\rho_1$  is *more robust than*  $\rho_2$ .

We immediately get the following corollary.

**Corollary 2.17.** *For two law-invariant convex risk measures  $\rho_1$  and  $\rho_2$  on  $L^\infty$ , the following conditions are equivalent.*

- (a)  $\rho_1$  is at least as robust as  $\rho_2$ .
- (b) When the Young function  $\Psi$  satisfies the  $\Delta_2$ -condition (2.3) and  $\bar{\rho}_2$  is finite on  $H^\Psi$ , then  $\bar{\rho}_1$  is also finite on  $H^\Psi$ .

Instead of comparing the robustness of two risk measures with respect to all possible weight functions  $\psi$ , it makes sense to restrict the attention to the weight functions  $\psi_p(x) = |x|^p/p$  for  $0 < p < \infty$ . This leads to the following notion that was first proposed in [22] within a more general context.

**Definition 2.18** (Index of qualitative robustness). Let  $\rho$  be a law-invariant convex risk measure on  $L^\infty$ . The associated *index of qualitative robustness* is defined as

$$\text{iqr}(\rho) = \left( \inf \{ p \in (0, \infty) : \mathcal{R}_\rho \text{ is } \psi_p\text{-robust on } \mathcal{M}(L^\infty) \} \right)^{-1}.$$

It follows from Proposition 2.13 that any law-invariant convex risk measure  $\rho$  satisfies  $\text{iqr}(\rho) \leq 1$ . Thus, Theorem 2.15 implies that

$$(2.14) \quad \text{iqr}(\rho) = \left( \inf \{p \in [1, \infty) : \bar{\rho} \text{ is finite on } L^p\} \right)^{-1}.$$

Our next result shows that the index of qualitative robustness is a convenient tool for comparing the qualitative robustness of two law-invariant convex risk measures.

**Theorem 2.19.** *Suppose that  $\rho_1, \rho_2$  are two law-invariant convex risk measures such that  $\text{iqr}(\rho_1) > \text{iqr}(\rho_2)$ . Then  $\rho_1$  is more robust than  $\rho_2$ .*

**Example 2.20** (Risk measure based on one-sided moments). For the risk measure  $\rho$  defined in (2.4) with  $p \in [1, \infty)$  and  $a > 0$  we obviously have  $\text{iqr}(\rho) = 1/p$ .  $\diamond$

## 2.5 Index of qualitative robustness for distortion risk measures

We now turn to the important example class of *distortion risk measures* defined as

$$(2.15) \quad \rho_g(X) := \int_{-\infty}^0 g(F_X(y)) dy - \int_0^{\infty} (1 - g(F_X(y))) dy,$$

where  $F_X$  denotes the distribution function of  $X$ , and  $g$  is a nondecreasing function such that  $g(0) = 0$  and  $g(1) = 1$ ; see e.g. [32, 13, 30, 23, 18]. It is a consequence of a theorem by Dellacherie and Schmeidler that  $\rho_g$  is a law-invariant convex risk measure on  $L^\infty$  if and only if  $g$  is concave (see Proposition 4.75 and Theorem 4.94 in [18] for a proof). In this case,  $\rho_g$  is even coherent and can be represented as

$$(2.16) \quad \rho_g(X) = g(0+) \text{ess sup}(-X) + \int_0^1 \text{V@R}_t(X) g'_+(t) dt, \quad X \in L^\infty,$$

where  $g'_+$  is the right-hand derivative of  $g$  and  $\text{V@R}_t(X) := -\inf\{y : F_X(y) > t\}$  denotes the Value at Risk at level  $t$ ; see, e.g., [18, Theorem 4.70]. It was pointed out in [9] that in this case  $\rho$  cannot be qualitatively 1l-robust. On the other hand, the risk functionals of two different concave distortion risk measures may have a rather different behavior in regard to qualitative robustness. This is the content of the following result.

**Proposition 2.21.** *Suppose that  $g$  is concave and continuous. Then, for  $p \in [1, \infty)$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , the extension  $\bar{\rho}_g$  of  $\rho_g$  is finite on  $L^p$  if and only if  $g'_+ \in L^q(0, 1)$ . In particular,*

$$\text{iqr}(\rho_g) = \frac{q^* - 1}{q^*} \quad \text{where} \quad q^* = \sup \left\{ q \geq 1 \mid \int_0^1 (g'_+(t))^q dt < \infty \right\}.$$

**Example 2.22** (Average Value at Risk). The risk measure Average Value at Risk at level  $\alpha \in (0, 1)$ ,  $\text{AV@R}_\alpha$ , is given in terms of the concave distortion function  $g_1(t) = (t/\alpha) \wedge 1$  (see [18, Example 4.71]).  $\text{AV@R}$  is also called Expected Shortfall, Conditional Value at Risk, or TailVaR. Since  $g'_1$  is bounded, it follows from Proposition 2.21 that  $\text{iqr}(\text{AV@R}_\alpha) = 1$ .

More generally, we can consider the distortion function  $g_\beta(t) = (t/\alpha)^\beta \wedge 1$  for some  $\beta \in (0, 1]$ . It is easy to see that the corresponding risk measure  $\rho_{g_\beta}$  has  $\text{iqr}(\rho_{g_\beta}) = \beta$ . It follows already from this simple example that distortion risk measures cover the whole possible range of our index of qualitative robustness.  $\diamond$

**Example 2.23** (MINMAXVAR). In [7, 8] the concave distortion risk measures MINVAR, MAXVAR, MINMAXVAR, and MAXMINVAR were introduced. They play an important role in conic finance [8]. For instance, MINMAXVAR is defined in terms of the concave distortion function

$$g_{\lambda,\gamma}(t) = 1 - (1 - t^{\frac{1}{1+\lambda}})^{1+\gamma},$$

where  $\lambda$  and  $\gamma$  are nonnegative parameters. An easy computation shows that  $g'_{\lambda,\gamma}(t) \sim c \cdot t^{-\frac{\lambda}{1+\lambda}}$  as  $t \downarrow 0$ , and so we have  $\text{iqr}(\text{MINMAXVAR}) = \frac{1}{1+\lambda}$ .  $\diamond$

### 3 Some general results

In this section, we will state and prove some theorems that are crucial ingredients for the proofs of the results from Section 2. These theorems and propositions are interesting in their own right and stated in the general contexts of robust statistics and  $\psi$ -weak convergence.

#### 3.1 Hampel-type theorems for the $\psi$ -weak topology

Hampel [19] introduced the notion of qualitative robustness with the weak topology in mind. In Section 2.4 we have argued that it is necessary to replace the weak topology with a finer  $\psi$ -weak topology so as to obtain a more balanced picture of the robustness of a risk functional. Such an approach was first suggested in [22]. In this section, we will give versions of Hampel's theorem and its converse for the  $\psi$ -weak topology that are slightly stronger than the corresponding results in [22]. We need them as basis for the results in Section 2.4, but we will state them here in the framework of robust statistics rather than in the narrower context of risk functionals. For the sake of consistency with the preceding sections, we have chosen a one-dimensional setting, but we could just as well have stated our results in the even more general multivariate framework of [22].

As in Section 2.4, we consider the canonical product space  $\Omega := \mathbb{R}^{\mathbb{N}}$  with Borel field  $\mathcal{F}$  and coordinate mappings  $(X_n)$ , which become i.i.d. random variables under a product measure  $\mathbb{P}_\mu = \mu^{\otimes \mathbb{N}}$ . A *statistical functional* will be a map  $T : \mathcal{M} \rightarrow \mathbb{R}$ , where  $\mathcal{M} \subset \mathcal{M}_1$  must contain all measures of the form  $\frac{1}{n} \sum_{k=1}^n \delta_{x_k}$  for  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in \mathbb{R}$ . It gives rise to a sequence of estimators given by  $\hat{T}_n := T(\hat{m}_n)$ , where  $\hat{m}_n$  is the empirical distribution of  $X_1, \dots, X_n$  as in (2.8). Clearly, the risk functional  $\mathcal{R}_\rho$  associated with a law-invariant risk measure  $\rho$  is an example of a statistical functional. Also, recall from Definition 2.11 the notion of a uniformly  $\psi$ -integrating set. The following definition is a modified version of [22, Definition 2.1].

**Definition 3.1** ( $\psi$ -robustness). Let  $T$  be a statistical functional and  $\mathcal{M}$  be a subset of  $\mathcal{M}_1^\psi$ . Then  $T$  is called  $\psi$ -robust at  $\mu$  in  $\mathcal{M}$  if for each  $\varepsilon > 0$  and every uniformly  $\psi$ -integrating set  $\mathcal{N} \subset \mathcal{M}$  with  $\mu \in \mathcal{N}$  there are  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that

$$\nu \in \mathcal{N}, d_\psi(\mu, \nu) \leq \delta \implies d_{\text{Proh}}(\mathbb{P}_\mu \circ \hat{T}_n^{-1}, \mathbb{P}_\nu \circ \hat{T}_n^{-1}) \leq \varepsilon \quad \text{for } n \geq n_0.$$

The following theorem provides a version of Hampel's theorem that is stronger than [22, Corollary 3.6], the corresponding result in [22].

**Theorem 3.2** (Hampel's theorem for the  $\psi$ -weak topology). *Let  $T : \mathcal{M} \rightarrow \mathbb{R}$  be a statistical functional where  $\mathcal{M} \subset \mathcal{M}_1^\psi$ . When  $T : \mathcal{M} \rightarrow \mathbb{R}$  is  $\psi$ -weakly continuous at  $\mu \in \mathcal{M}$ , then  $T$  is  $\psi$ -robust at  $\mu$  in  $\mathcal{M}$ .*

*Proof.* The result will follow from [22, Theorem 2.4] when we can show that every uniformly  $\psi$ -integrating set  $\mathcal{N}$  has the following *uniform Glivenko–Cantelli (UGC) property*: for each  $\varepsilon > 0$  there are  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that

$$(3.1) \quad \sup_{\nu \in \mathcal{N}} \mathbb{P}_\nu [d_\psi(\nu, \widehat{m}_n) \geq \delta] \leq \varepsilon \quad \text{for } n \geq n_0.$$

According to [24, Lemma 4], the set  $\mathcal{M}_1$  has the UGC property for the Prohorov metric. Therefore, the UGC property (3.1) follows from (2.12) and Chung’s uniform weak law of large numbers ([10]; see also [28, Proposition A.5.1]) applied to the sequence  $(\psi(X_n))$  of random variables; notice that  $\int \psi d\widehat{m}_n = \frac{1}{n} \sum_{i=1}^n \psi(X_i)$  and  $\int \psi d\nu = \mathbb{E}_\nu[\psi(X_1)]$ .  $\square$

The following result may be viewed as a converse of Hampel’s theorem for the  $\psi$ -weak topology. Together with Theorem 3.2 and with the choice  $\psi \equiv 1$  and  $\mathcal{M} = \mathcal{M}_1$  it yields the classical Hampel theorem in the form of [20, Theorem 2.21]. Its statement uses the following notion of consistency: a statistical functional  $T$  is called *weakly consistent at  $\mu \in \mathcal{M}$*  when  $\widehat{T}_n \rightarrow T(\mu)$  in  $\mathbb{P}_\mu$ -probability.

**Theorem 3.3** (Converse of Hampel’s theorem for the  $\psi$ -weak topology). *Suppose that  $T : \mathcal{M} \rightarrow \mathbb{R}$  is a statistical functional where  $\mathcal{M} \subset \mathcal{M}_1^\psi$ . Let  $\mu \in \mathcal{M}$  and  $\delta_0 > 0$  be given, and suppose that  $T$  is weakly consistent at each  $\nu$  in  $\mathcal{M}$  with  $d_\psi(\nu, \mu) \leq \delta_0$ . When  $T$  is  $\psi$ -robust at  $\mu$  in  $\mathcal{M}$ , then  $T : \mathcal{M} \rightarrow \mathbb{R}$  is  $\psi$ -weakly continuous at  $\mu$ .*

*Proof.* We must show that  $T(\mu_k) \rightarrow T(\mu)$  when  $(\mu_k)$  is a sequence in  $\mathcal{M}$  that converges  $\psi$ -weakly to  $\mu$ . Given such a sequence  $(\mu_k)$ , the set  $\mathcal{N} := \{\mu, \mu_1, \mu_2, \dots\}$  is clearly compact for the  $\psi$ -weak topology. By Lemma A.4 we conclude that  $\mathcal{N}$  is uniformly  $\psi$ -integrating. So, given  $\varepsilon > 0$ , the  $\psi$ -robustness of  $T$  at  $\mu$  in  $\mathcal{M}$  implies that there are some  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that  $d_{\text{Proh}}(\mathbb{P}_{\mu_k} \circ \widehat{T}_n^{-1}, \mathbb{P}_\mu \circ \widehat{T}_n^{-1}) \leq \varepsilon$  for all  $n \geq n_0$  and  $k \geq k_0$ , where  $k_0 \in \mathbb{N}$  is chosen such that  $d_\psi(\mu_k, \mu) \leq \delta$  for all  $k \geq k_0$  (recall that  $d_\psi$  generates the  $\psi$ -weak topology). So, assuming without loss of generality  $\delta \leq \delta_0$ , the weak consistency of  $(\widehat{T}_n)$  at  $\mu_k$  implies that

$$\begin{aligned} |T(\mu_k) - T(\mu)| &= d_{\text{Proh}}(\delta_{T(\mu_k)}, \delta_{T(\mu)}) \\ &\leq \limsup_{n \rightarrow \infty} \left( d_{\text{Proh}}(\delta_{T(\mu_k)}, \mathbb{P}_{\mu_k} \circ \widehat{T}_n^{-1}) + \varepsilon + d_{\text{Proh}}(\mathbb{P}_\mu \circ \widehat{T}_n^{-1}, \delta_{T(\mu)}) \right) \\ &= \varepsilon \end{aligned}$$

for all  $k \geq k_0$ . This completes the proof.  $\square$

### 3.2 Skorohod representation for $\psi$ -weak convergence

The classical Skorohod–Dudley–Wichura representation theorem states that weak convergence  $\mu_n \rightarrow \mu_0$  is equivalent to the existence of random variables  $X_n$  with law  $\mu_n$  such that  $X_n \rightarrow X_0$  almost surely. A question one may ask is whether  $\psi$ -weak convergence  $\mu_n \rightarrow \mu_0$  can be expressed in terms of a stronger concept for the convergence  $X_n \rightarrow X_0$ . Here we are going to address this question in the context of the Orlicz spaces.

**Theorem 3.4.** *For any finite Young function  $\Psi$  the following two conditions are equivalent.*

- (a) *A sequence  $(\mu_n)$  in  $\mathcal{M}(H^\Psi)$  converges  $\Psi$ -weakly to some  $\mu_0$  if and only if there exists a sequence  $(X_n)_{n \in \mathbb{N}_0}$  in  $H^\Psi$  such that  $X_n$  has law  $\mu_n$  for each  $n$  and  $\|X_n - X_0\|_\Psi \rightarrow 0$ .*

(b)  $\Psi$  satisfies the  $\Delta_2$ -condition (2.3).

For proving Theorem 3.4 we need the following lemma.

**Lemma 3.5.** *Let  $\Psi$  be a Young function satisfying the  $\Delta_2$ -condition (2.3) and let  $(X_n)_{n \in \mathbb{N}_0}$  be a sequence in  $H^\Psi = L^\Psi$ . If the sequence  $(\Psi(|X_n|))_{n \in \mathbb{N}_0}$  is uniformly integrable, then the sequence  $(\Psi(2^m |X_n - X_0|))_{n \in \mathbb{N}}$  is also uniformly integrable for every  $m \in \mathbb{N}_0$ .*

*Proof.* By (2.3), we have  $C := \sup_{x \geq x_0} \Psi(2x)/\Psi(x) < \infty$  for some  $x_0 > 0$ . We proceed by induction on  $m \in \mathbb{N}_0$ .

First, let  $m = 0$ . Since  $\Psi$  is nondecreasing and convex with  $\Psi(0) = 0$ , we obtain by the triangle inequality

$$\begin{aligned} \Psi(|X_n - X_0|) &\leq \frac{1}{2} \left( \Psi(2|X_n|) + \Psi(2|X_0|) \right) \\ &= \frac{1}{2} \left( \Psi(2 \mathbb{1}_{[0, x_0]}(|X_n|)|X_n|) + \Psi(2 \mathbb{1}_{[0, x_0]}(|X_0|)|X_0|) \right) \\ &\quad + \frac{1}{2} \left( \Psi(2 \mathbb{1}_{(x_0, \infty)}(|X_n|)|X_n|) + \Psi(2 \mathbb{1}_{(x_0, \infty)}(|X_0|)|X_0|) \right) \\ &\leq \Psi(2x_0) + \frac{C}{2} \left( \Psi(|X_n|) + \Psi(|X_0|) \right). \end{aligned}$$

Since the sequence  $(\Psi(|X_n|))_{n \in \mathbb{N}_0}$  is uniformly integrable by assumption, we may thus conclude that the sequence  $(\Psi(|X_n - X_0|))_{n \in \mathbb{N}}$  is uniformly integrable.

Let us now suppose that  $(\Psi(2^m |X_n - X_0|))_{n \in \mathbb{N}}$  is uniformly integrable for any given  $m \in \mathbb{N}_0$ . Following an analogous line of reasoning as in the case of  $m = 0$ , we may find

$$\Psi(2^{m+1} |X_n - X_0|) = \Psi(2 \cdot 2^m |X_n - X_0|) \leq \Psi(2x_0) + C \Psi(2^m |X_n - X_0|).$$

Hence,  $(\Psi(2^{m+1} |X_n - X_0|))_{n \in \mathbb{N}}$  is uniformly integrable, which completes the proof.  $\square$

*Proof of (b)  $\Rightarrow$  (a) in Theorem 3.4.* Let us suppose that the  $\Delta_2$ -condition (2.3) holds.

We first prove that  $\|X_n - X_0\|_\Psi \rightarrow 0$  implies  $\mu_n \rightarrow \mu_0$   $\Psi$ -weakly. Due to Proposition 2.1.10 in [14],  $\|X_n - X_0\|_\Psi \rightarrow 0$  yields  $\mathbb{E}[\Psi(2|X_n - X_0|)] \rightarrow 0$  and  $X_n \rightarrow X_0$  in probability. Convexity and monotonicity of  $\Psi$  imply that

$$0 \leq \Psi(|X_n|) \leq \frac{1}{2} \Psi(2||X_n| - |X_0||) + \frac{1}{2} \Psi(2|X_0|) \leq \frac{1}{2} \Psi(2|X_n - X_0|) + \frac{1}{2} \Psi(2|X_0|).$$

Hence, dominated convergence yields that

$$\int \Psi(|x|) \mu_n(dx) = \mathbb{E}[\Psi(|X_n|)] \longrightarrow \mathbb{E}[\Psi(|X_0|)] = \int \Psi(|x|) \mu_0(dx).$$

Moreover, since  $X_n \rightarrow X_0$  in probability the corresponding laws  $(\mu_n)$  converge weakly. Now the  $\Psi$ -weak convergence  $\mu_n \rightarrow \mu_0$  follows from Lemma A.3 (iv)  $\Rightarrow$  (i).

Now we prove that the  $\Psi$ -weak convergence  $\mu_n \rightarrow \mu_0$  implies the existence of a sequence  $(X_n)$  in  $H^\Psi$  such that  $\|X_n - X_0\|_\Psi \rightarrow 0$ . Clearly,  $\mu_n \rightarrow \mu_0$  weakly. By Skorohod representation there



hence exists a sequence of random variables  $(X_n)$  such that  $X_n \rightarrow X_0$   $\mathbb{P}$ -a.s. The continuity of  $\Psi$  and the fact that  $\Psi(0) = 0$  yield that

$$(3.2) \quad \Psi(|X_n|) \rightarrow \Psi(|X_0|) \quad \mathbb{P}\text{-a.s.}$$

$$(3.3) \quad \Psi(k|X_n - X_0|) \rightarrow 0 \quad \mathbb{P}\text{-a.s. for all } k \geq 0.$$

Moreover, the  $\Psi$ -weak convergence  $\mu_n \rightarrow \mu_0$  implies that

$$(3.4) \quad \mathbb{E}[\Psi(|X_n|)] = \int \Psi(|x|) \mu_n(dx) \rightarrow \int \Psi(|x|) \mu_0(dx) = \mathbb{E}[\Psi(|X_0|)].$$

Now, (3.2), (3.4), and Vitali's theorem in the form of [21, Proposition 3.12 (ii) $\Rightarrow$ (iii)] imply that the sequence  $(\Psi(|X_n|))_{n \in \mathbb{N}_0}$  is uniformly integrable. Applying Lemma 3.5 yields the uniform integrability of the sequence  $(\Psi(k|X_n - X_0|))_{n \in \mathbb{N}}$  for every  $k > 0$ . Therefore, (3.3) and another application of Vitali's theorem, this time in the form of [21, Proposition 3.12 (iii) $\Rightarrow$ (ii)], yield  $\mathbb{E}[\Psi(k|X_n - X_0|)] \rightarrow 0$  for every  $k > 0$ , which implies  $\|X_n - X_0\|_\Psi \rightarrow 0$  according to [14, Proposition 2.1.10 (5)]. Finally, the sequence  $(X_n)$  belongs to  $H^\Psi$ , because under the  $\Delta_2$ -condition  $H^\Psi$  coincides with the class of random variables  $Y$  with  $\mathbb{E}[\Psi(|Y|)] < \infty$ .  $\square$

*Proof of (a) $\Rightarrow$ (b) in Theorem 3.4.* Let us suppose that condition (a) in Theorem 3.4 holds, but that  $\Psi$  does not satisfy the  $\Delta_2$ -condition (2.3). We will show that this leads to a contradiction. Since  $\Psi$  does not satisfy the  $\Delta_2$ -condition (2.3) and our probability space is atomless, we have  $H^\Psi \neq L^\Psi$  by [14, Theorem 2.1.17]. Hence there exists a random variable  $Y \geq 0$  such that  $\mathbb{E}[\Psi(Y)] < \infty$  and  $\mathbb{E}[\Psi(2Y)] = \infty$ . We then choose  $a_n > 0$  such that

$$2\mathbb{E}[\Psi(2(Y \wedge a_n))] \geq n + \Psi(4n) \quad \text{for each } n,$$

and let

$$X_n := (Y - n)^+ \wedge a_n.$$

Then  $X_n \in L^\infty$ , and hence  $X_n \in H^\Psi$  since  $\Psi$  is finite. Moreover,  $X_n \rightarrow X_0 := 0$ , and

$$(3.5) \quad 0 \leq \mathbb{E}[\Psi(|X_n|)] = \mathbb{E}[\Psi((Y - n)^+ \wedge a_n)] \leq \mathbb{E}[\Psi(Y - n)] \rightarrow 0$$

by dominated convergence. It therefore follows from Lemma A.3 that  $\mu_n := \mathbb{P} \circ X_n^{-1} \rightarrow \delta_0$  in the  $\Psi$ -weak topology.

We will show next that we cannot have  $\|X_n - X_0\|_\Psi = \|X_n\|_\Psi \rightarrow 0$ . Since any sequence  $(\tilde{X}_n)$  for which  $\mathbb{P} \circ \tilde{X}_n^{-1} = \mu_n$  must satisfy  $\|\tilde{X}_n\|_\Psi = \|X_n\|_\Psi$ , condition (a) in Theorem 3.4 will thus be violated. So let us suppose by way of contradiction that  $\|X_n\|_\Psi \rightarrow 0$ . By [14, Proposition 2.1.10 (5)], this is equivalent to  $\mathbb{E}[\Psi(k|X_n|)] \rightarrow 0$  for every  $k > 0$ . By taking  $k = 4$  and using the fact that the convex function  $\ell(x) := \Psi(4(0 \vee x))$  satisfies  $\ell(x - y) \geq 2\ell(x/2) - \ell(y)$  we obtain

$$\begin{aligned} \mathbb{E}[\Psi(4|X_n|)] &= \mathbb{E}[\Psi(4((Y - n)^+ \wedge a_n))] \geq \mathbb{E}[\Psi(4(Y \wedge (a_n + n) - n)^+)] \\ &\geq 2\mathbb{E}[\Psi(2(Y \wedge (a_n + n)))] - \Psi(4n) \\ &\geq 2\mathbb{E}[\Psi(2(Y \wedge a_n))] - \Psi(4n) \geq n, \end{aligned}$$

by construction. This is the desired contradiction.  $\square$

## 4 Proofs of the results from Section 2

*Proof of Theorem 2.5.* Since  $\mathbb{E}[\Psi(k|X|)] < \infty$  for each  $k > 0$ , Birkhoff's ergodic theorem (e.g., in the form of [5, Theorem 6.28]) implies that for each  $k > 0$  and  $\mathbb{P}$ -a.e.  $\omega \in \Omega$

$$(4.1) \quad \int \Psi(k|x|) \widehat{m}_n(\omega)(dx) = \frac{1}{n} \sum_{i=1}^n \Psi(k|X_i(\omega)|) \longrightarrow \mathbb{E}[\Psi(k|X|)] = \int \Psi(k|x|) \mu(dx),$$

where  $\mu := \mathbb{P} \circ X^{-1}$ . Moreover, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$

$$(4.2) \quad \widehat{m}_n(\omega) \longrightarrow \mu \quad \text{weakly,}$$

due to Birkhoff's ergodic theorem and an application of [25, Theorem 6.6]. Hence there exists a measurable set  $\Omega_0 \in \mathcal{F}$  such that  $\mathbb{P}[\Omega_0] = 1$  and such that for each  $\omega \in \Omega_0$  (4.2) is satisfied and (4.1) holds for each  $k \in \mathbb{N}$ . Let us fix  $\omega_0 \in \Omega_0$ . Since our probability space is atomless, standard Skorohod representation yields the existence of random variables  $X^{\omega_0}$ ,  $(X_n^{\omega_0})$  such that  $X^{\omega_0}$  has law  $\mu$ ,  $X_n^{\omega_0}$  has law  $\widehat{m}_n(\omega_0)$ , and  $X_n^{\omega_0} \rightarrow X^{\omega_0}$   $\mathbb{P}$ -a.s. By (4.1), for each  $k \in \mathbb{N}$ ,

$$(4.3) \quad \mathbb{E}[\Psi(k|X_n^{\omega_0}|)] = \int \Psi(k|x|) \widehat{m}_n(\omega_0)(dx) \longrightarrow \int \Psi(k|x|) \mu(dx) = \mathbb{E}[\Psi(k|X^{\omega_0}|)].$$

Therefore the sequence  $(\Psi(k|X_n^{\omega_0}|))$  is uniformly integrable for each  $k \in \mathbb{N}$ .

Now take  $a > 0$  and pick  $k \in \mathbb{N}$  such that  $k \geq 2a$ . Since  $\Psi$  is convex and nondecreasing, we have

$$0 \leq \Psi(a|X_n^{\omega_0} - X^{\omega_0}|) \leq \frac{1}{2} \left( \Psi(2a|X_n^{\omega_0}|) + \Psi(2a|X^{\omega_0}|) \right) \leq \frac{1}{2} \left( \Psi(k|X_n^{\omega_0}|) + \Psi(k|X^{\omega_0}|) \right).$$

It follows that the sequence  $(\Psi(a|X_n^{\omega_0} - X^{\omega_0}|))$  is uniformly integrable. Since clearly  $\Psi(a|X_n^{\omega_0} - X^{\omega_0}|) \rightarrow 0$   $\mathbb{P}$ -a.s., we get that  $\mathbb{E}[\Psi(a|X_n^{\omega_0} - X^{\omega_0}|)] \rightarrow 0$  for each  $a > 0$  and in turn that  $\|X_n^{\omega_0} - X^{\omega_0}\|_{\Psi} \rightarrow 0$  (see [14, Proposition 2.1.10]). By [6, Theorem 4.1]  $\rho$  is continuous with respect to the Luxemburg norm  $\|\cdot\|_{\Psi}$ , and so

$$\widehat{\rho}_n(\omega_0) = \mathcal{R}_{\rho}(\widehat{m}_n(\omega_0)) = \rho(X_n^{\omega_0}) \longrightarrow \rho(X^{\omega_0}) = \rho(X)$$

for each  $\omega_0 \in \Omega_0$ . □

*Proof of Theorem 2.7.* We first prove the implication (b) $\Rightarrow$ (a) in Theorem 2.7. So let us assume that  $\Psi$  satisfies the  $\Delta_2$ -condition (2.3) and let  $\rho$  be a convex risk measure on  $H^{\Psi}$  with associated map  $\mathcal{R}_{\rho}$ . To show the continuity of  $\mathcal{R}_{\rho}$ , choose a sequence  $(\mu_n)$  such that  $\mu_n \rightarrow \mu$   $\Psi$ -weakly. By Theorem 3.4 there exists a sequence  $(X_n)$  in  $H^{\Psi}$  such that each  $X_n$  has law  $\mu_n$  and such that  $\|X_n - X_0\|_{\Psi} \rightarrow 0$ . But it was shown in [6, Theorem 4.1] that  $\rho$  is continuous with respect to the Luxemburg norm  $\|\cdot\|_{\Psi}$  (see, also [26, Proposition 3.1]). Therefore,

$$\mathcal{R}_{\rho}(\mu_n) = \rho(X_n) \longrightarrow \rho(X_0) = \mathcal{R}_{\rho}(\mu_0),$$

which proves the implication (b) $\Rightarrow$ (a).

We now prove the implication (a) $\Rightarrow$ (b) in Theorem 2.7. This proof is similar to the proof of (a) $\Rightarrow$ (b) in Theorem 3.4. We assume that  $\Psi$  does not satisfy the  $\Delta_2$ -condition (2.3), and we will construct a risk measure  $\rho$  for which  $\mathcal{R}_{\rho}$  is not  $\Psi$ -weakly continuous. This risk measure is given

as the utility-based shortfall risk measure (2.6) with convex loss function  $\ell(x) := \Psi(0 \vee (8x))$ . It follows as in (2.7) that  $\mathbb{E}[\ell(-X - m)]$  is finite and well-defined for  $m \in \mathbb{R}$  and  $X \in H^\Psi$ .

Since  $\Psi$  does not satisfy the  $\Delta_2$ -condition (2.3) and our probability space is atomless, we have  $H^\Psi \neq L^\Psi$  by [14, Theorem 2.1.17]. Hence there exists a random variable  $Y \geq 0$  such that  $\mathbb{E}[\Psi(Y)] < \infty$  and  $\mathbb{E}[\Psi(2Y)] = \infty$ . We then choose  $a_n > 0$  such that

$$(4.4) \quad 4\mathbb{E}\left[\Psi(2(Y \wedge a_n))\right] \geq n + \ell(n/2) \quad \text{for each } n,$$

and let

$$X_n := -((Y - n)^+ \wedge a_n).$$

Then  $X_n \in L^\infty$  and hence  $X_n \in H^\Psi$  since  $\Psi$  is finite. As in (3.5) we get  $\mathbb{E}[\Psi(|X_n|)] \rightarrow 0$ , and so  $\mu_n := \mathbb{P} \circ X_n^{-1} \rightarrow \delta_0$  in the  $\Psi$ -weak topology. We now prove that the sequence  $z_n := \rho(X_n) = \mathcal{R}_\rho(\mu_n)$  is unbounded, which will imply that  $\mathcal{R}_\rho$  is not continuous for the  $\Psi$ -weak topology.

To prove that the sequence  $(z_n)$  is unbounded, we assume by way of contradiction that  $z^* := \sup_n z_n < \infty$ . We see from (2.7) and dominated convergence that each  $z_n = \rho(X_n)$  solves the equation  $\mathbb{E}[\ell(-X_n - z_n)] = 1$ . The convexity of  $\ell$  implies that  $\ell(x - y) \geq 2\ell(x/2) - \ell(y)$ . Hence,

$$\begin{aligned} 1 &= \mathbb{E}[\ell(-X_n - z_n)] \geq \mathbb{E}[\ell(-X_n - z^*)] \geq 2\mathbb{E}\left[\ell\left(\frac{1}{2}((Y - n)^+ \wedge a_n)\right)\right] - \ell(z^*) \\ &\geq 2\mathbb{E}\left[\ell\left(\frac{1}{2}(Y \wedge (a_n + n)) - \frac{n}{2}\right)\right] - \ell(z^*) \\ &\geq 4\mathbb{E}\left[\ell\left(\frac{1}{4}(Y \wedge a_n)\right)\right] - \ell(n/2) - \ell(z^*) \\ &= 4\mathbb{E}\left[\Psi(2(Y \wedge a_n))\right] - \ell(n/2) - \ell(z^*). \end{aligned}$$

But according to (4.4), the expression on the right is bounded from below by  $n - \ell(2z^*)$ , which yields the desired contradiction.  $\square$

*Proof of Theorem 2.9.* The equivalence between conditions (a) and (b) follows from Theorem 2.7. The implication (b) $\Rightarrow$ (c) simply follows from the fact that  $\rho$  is equal to the restriction of  $\bar{\rho}$  to  $L^\infty$ .

To prove (c) $\Rightarrow$ (d), we first note that  $\mu_n := \mathbb{P} \circ X_n \in \mathcal{M}(L^\infty)$  when  $(X_n)$  is a sequence as in (d). Moreover,  $\|X_n\|_\Psi \rightarrow 0$  implies that  $\mu_n \rightarrow \delta_0$   $\Psi$ -weakly. So it is now clear that (c) implies (d).

We now prove (d) $\Rightarrow$ (a). To this end, we will apply [6, Theorem 4.3], which states that  $\bar{\rho}$  is finite on  $H^\Psi$  when 0 belongs to the topological interior of the effective domain of the map  $\bar{\rho} : H^\Psi \rightarrow \mathbb{R} \cup \{+\infty\}$ . We will therefore show that  $\rho$  is finite on the centered  $\varepsilon$ -ball  $B_\varepsilon := \{X \in H^\Psi : \|X\|_\Psi < \varepsilon\}$  when  $\varepsilon > 0$  is small enough.

Suppose that  $(X_n)$  is a sequence in  $L^\infty$  such that  $\|X_n\|_\Psi \rightarrow 0$ . Then  $\mu_n := \mathbb{P} \circ X_n^{-1} \rightarrow \delta_0$   $\Psi$ -weakly, and so  $\rho(X_n) = \mathcal{R}_\rho(\mu_n) \rightarrow \mathcal{R}_\rho(\delta_0) = \rho(0)$ . Thus,  $\rho : L^\infty \rightarrow \mathbb{R}$  is continuous with respect to  $\|\cdot\|_\Psi$  at 0. Hence, for  $K > 0$  given, there exists  $\varepsilon > 0$  such that  $\rho(X) \leq K$  for  $X \in B_\varepsilon \cap L^\infty$ . Now let us fix  $X \in B_\varepsilon$ . The negative part  $X^-$  belongs again to  $B_\varepsilon$ , and monotone convergence yields  $X^- \wedge k \rightarrow X^-$  in  $L^1$ . Using the lower semicontinuity of  $\bar{\rho} : L^1 \rightarrow \mathbb{R} \cup \{+\infty\}$  hence gives,

$$\bar{\rho}(X) \leq \bar{\rho}(-X^-) \leq \liminf_{k \uparrow \infty} \bar{\rho}(-X^- \wedge k) = \liminf_{k \uparrow \infty} \rho(-X^- \wedge k) \leq K.$$

Here we have also used the monotonicity of  $\bar{\rho}$  in the first and the fact that  $-(X^- \wedge k) \in B_\varepsilon \cap L^\infty$  in the final step.  $\square$

*Proof of Proposition 2.13.* We prove the assertion by way of contradiction. So let  $\psi : \mathbb{R}_+ \rightarrow (0, \infty)$  be a nondecreasing function such that  $\liminf_{x \uparrow \infty} \psi(x)/x = 0$  and suppose that  $\mathcal{R}_\rho$  is  $\psi(|\cdot|)$ -robust on  $\mathcal{M}(L^\infty)$ . By Theorem 2.5,  $\mathcal{R}_\rho$  is consistent at each  $\mu \in \mathcal{M}(L^\infty)$ . Hence Theorem 3.3 and the robustness of  $\mathcal{R}_\rho$  imply the continuity of  $\mathcal{R}_\rho$  on  $\mathcal{M}(L^\infty)$  with respect to  $\psi(|\cdot|)$ -weak convergence. As in the proofs of Theorems 3.4 and 2.7 we will construct a sequence  $(\mu_n) \subset \mathcal{M}(L^\infty)$  that converges  $\psi(|\cdot|)$ -weakly to  $\delta_0$  but for which  $\mathcal{R}_\rho(\mu_n) \not\rightarrow \mathcal{R}_\rho(\delta_0)$ . To this end, we easily construct a random variable  $Y \geq 0$  such that  $\mathbb{E}[\psi(Y)] < \infty$  and  $\mathbb{E}[Y] = \infty$  and pick  $a_n > 0$  such that  $\mathbb{E}[Y \wedge a_n] \geq 2n$ . Then  $X_n := (Y - n)^+ \wedge a_n \rightarrow 0$   $\mathbb{P}$ -a.s. and  $\mathbb{E}[\psi(X_n)] \rightarrow 0$  by dominated convergence. Hence,  $\mu_n := \mathbb{P} \circ X_n^{-1}$  converge  $\psi(|\cdot|)$ -weakly to  $\delta_0$  by Lemma A.3. However,  $\mathbb{E}[X_n] \geq \mathbb{E}[Y \wedge a_n] - n \geq n$ . Now [27, Lemma 2.3] yields that

$$\mathcal{R}_\rho(\mu_n) = \rho(X_n) \geq \rho(\mathbb{E}[X_n]) \geq \rho(n),$$

which shows that we cannot have  $\mathcal{R}_\rho(\mu_n) \rightarrow \mathcal{R}_\rho(\delta_0)$ .  $\square$

*Proof of Theorem 2.14.* (a) $\Rightarrow$ (b): By Theorem 2.5,  $\mathcal{R}_\rho$  is consistent at each  $\mu \in \mathcal{M}(H^\Psi)$ . Hence Theorem 3.3 and the robustness of  $\mathcal{R}_\rho$  imply the continuity of  $\mathcal{R}_\rho$  on  $\mathcal{M}(H^\Psi)$  with respect to  $\Psi$ -weak convergence. Thus, due to Theorem 2.7,  $\Psi$  must satisfy the  $\Delta_2$ -condition (2.3).

(b) $\Rightarrow$ (a): By Theorem 2.7,  $\mathcal{R}_\rho$  is a continuous map on  $\mathcal{M}(H^\Psi) = \mathcal{M}_1^\Psi$ . Its  $\Psi$ -robustness on  $\mathcal{M}_1^\Psi$  therefore follows from Theorem 3.2.  $\square$

*Proof of Theorem 2.15.* The implication (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c): First, we note again that  $\mathcal{R}_\rho$  is consistent on  $\mathcal{M}(L^\infty)$  by Theorem 2.5. Therefore, Theorem 3.3 and the robustness of  $\mathcal{R}_\rho$  imply the  $\Psi$ -weak continuity of  $\mathcal{R}_\rho$  on  $\mathcal{M}(L^\infty)$ . Theorem 2.9 now yields (c).

(c) $\Rightarrow$ (a): Condition (c) implies that  $\bar{\rho}$  is a convex risk measure on  $H^\Psi$ . Hence, (a) follows from an application of Theorem 2.14.  $\square$

*Proof of Theorem 2.19.* Obviously, it suffices to show that  $\rho_1$  is at least robust as  $\rho_2$ . For this purpose let  $\Psi$  denote any finite Young function satisfying the  $\Delta_2$ -condition (2.3) such that  $\bar{\rho}_2$  is finite on  $H^\Psi$ .

In view of Lemma A.1 and Lemma A.2 (cf. Appendix A.1) we find some real number  $p_\Psi \geq 1$  such that  $L^p \subset H^\Psi$  is valid for  $p > p_\Psi$ , and  $H^\Psi \subset L^q$  holds for every  $q \in [1, p_\Psi)$ . Since  $\bar{\rho}_2$  is finite on  $H^\Psi$ , we may conclude

$$\text{iqr}(\rho_1) > \text{iqr}(\rho_2) \geq \frac{1}{p_\Psi}.$$

In particular there exists some  $1 \leq q < p_\Psi$  such that  $\bar{\rho}_1$  is finite on  $L^q$ . This implies that  $\bar{\rho}_1$  is finite on  $H^\Psi$  too because  $H^\Psi \subset L^q$  by choice of  $p_\Psi$ . Hence  $\rho_1$  is at least robust as  $\rho_2$  due to Corollary 2.17, and the proof is complete.  $\square$

*Proof of Proposition 2.21.* First, when  $g$  is continuous we have  $g(0+) = 0$  and in the ‘spectral’ representation (2.16) the part containing the essential supremum vanishes. Moreover, it follows from [16] that (2.16) remains true for  $\bar{\rho}_g$  and  $X \in L^1$ . Next, the function  $f(t) := g'_+(1-t)$  is nondecreasing, and we have

$$\int_0^1 V @ R_t(X) g'_+(t) dt = \int_0^1 q_{-X}(t) f(t) dt,$$

where  $q_{-X}$  is a quantile function for  $-X$ .

Now suppose that  $f \in L^q(0, 1)$ . By Hölder's inequality, we get that for any  $X \in L^p$ ,

$$\bar{\rho}_g(X) = \int_0^1 q_{-X}(t)f(t) dt \leq \left( \int_0^1 |q_{-X}(t)|^p dt \right)^{1/p} \left( \int_0^1 f(t)^q dt \right)^{1/q} = \mathbb{E}[|X|^p]^{1/p} \|f\|_{L^q} < \infty.$$

Here we have used [18, Lemma A.23] in the third step.

Now we assume that  $\bar{\rho}_g$  is finite on  $L^p$ . Since our probability space is atomless, it supports a random variable  $U$  with uniform distribution on  $(0, 1)$ . We will show that  $Y := f(U)$  belongs to  $L^q$ . According to the Banach–Steinhaus theorem this holds if and only if  $\mathbb{E}[(-X)Y] < \infty$  for all  $X \in L^p$ . But for  $X \in L^p$ , the fact that  $f$  is a quantile function for  $Y$  and the upper Hardy–Littlewood inequality (e.g., [18, Theorem A.24]) imply that

$$\infty > \rho(X) = \int_0^1 q_{-X}(t)f(t) dt = \int_0^1 q_{-X}(t)q_Y(t) dt \geq \mathbb{E}[(-X)Y].$$

This concludes the proof.  $\square$

## A Appendix

### A.1 Relation between Orlicz space and $L^p$ -space

Let  $\Psi$  be a Young function satisfying the  $\Delta_2$ -condition (2.3). In this section, we discuss some relations between the Orlicz space  $L^\Psi = H^\Psi$  and the standard  $L^p$ -spaces.

By the  $\Delta_2$ -condition (2.3),

$$\alpha(x) := \sup_{y \geq x} \frac{\Psi(2y)}{\Psi(y)}, \quad x \geq 0,$$

is a finite and nonincreasing function. Moreover,  $\alpha \geq 2$ , because the convexity of  $\Psi$  implies that  $\Psi(2x) \leq 2\Psi(x)$ . Hence we can define  $\alpha(\infty) := \lim_{x \uparrow \infty} \alpha(x) \in [2, \infty)$ .

**Lemma A.1.** *If*

$$\frac{2^p}{\alpha(x_0)} \geq \frac{\Psi(x_0)}{x_0^p}$$

*for some  $x_0 \in \alpha^{-1}(\mathbb{R})$ , then  $L^p \subset H^\Psi$ .*

*Proof.* Let  $x_0 \in \alpha^{-1}(\mathbb{R})$  such that

$$\frac{2^p}{\alpha(x_0)} \geq \frac{\Psi(x_0)}{x_0^p},$$

It is easy to check by induction that

$$(A.1) \quad (2^k x_0)^p \geq \Psi(2^k x_0) \quad \text{for } k \in \mathbb{N}.$$

Now, for any  $x \geq 2x_0$ , there exists a unique  $k \in \mathbb{N}$  with  $k \geq 2$  and  $2^{k-1}x_0 \leq x < 2^k x_0$ . Hence, by (A.1) and the monotonicity of  $\Psi$ ,

$$2^p x^p \geq (2^k x_0)^p \geq \Psi(2^k x_0) \geq \Psi(x).$$

Therefore,  $2^p x^p \geq \Psi(x)$  for  $x \geq 2x_0$ , and so [14, Theorem 2.2.3] implies  $L^p \subset L^\Psi = H^\Psi$ .  $\square$

In view of Lemma A.1 we have  $L^p \subset H^\Psi$  for at least one real number  $p \geq 1$ . Furthermore, we may define the number

$$p_\Psi := \inf\{p \geq 1 \mid L^p \subset H^\Psi\}.$$

The following Lemma considers the relationship between  $H^\Psi$  and  $L^p$  for  $1 \leq p < p_\Psi$ .

**Lemma A.2.** *The relationship  $H^\Psi \subset L^p$  holds for any  $p \in [1, p_\Psi)$ .*

*Proof.* Fix any  $p \in [1, p_\Psi)$ . By definition of  $p_\Psi$  we have  $L^p \not\subset H^\Psi$  so that in view of Lemma A.1 we may conclude

$$\frac{2^p}{\alpha(x_0)} < \frac{\Psi(x_0)}{x_0^p}$$

for every  $x_0 \in \alpha^{-1}(\mathbb{R})$ . Hence, for any strictly increasing, unbounded sequence  $(x_n)_n$  in  $\alpha^{-1}(\mathbb{R})$ , we obtain

$$0 < \frac{2^p}{\alpha(\infty)} \leq \liminf_{n \rightarrow \infty} \frac{2^p}{\alpha(x_n)} \leq \liminf_{n \rightarrow \infty} \frac{\Psi(x_n)}{x_n^p}.$$

This means

$$\beta(x_0) := \inf_{x \geq x_0} \frac{\Psi(x)}{x^p} > 0$$

for some  $x_0 \in \alpha^{-1}(\mathbb{R})$ , implying  $\Psi(x) \geq \beta(x_0)x^p$  for  $x \geq x_0$ . Then  $H^\Psi \subset L^p$  follows from [14, Theorem 2.2.3].  $\square$

## A.2 Auxiliary results on the $\psi$ -weak topology

First recall from [18, Corollary A.45] that the  $\psi$ -weak topology on  $\mathcal{M}_1^\psi(\mathbb{R})$  is separable and metrizable. The following lemma provides some useful characterizations of the  $\psi$ -weak convergence; see [22, Lemma 3.4] for a proof.

**Lemma A.3.** *The following statements are equivalent:*

- (i)  $\mu_n \rightarrow \mu$   $\psi$ -weakly.
- (ii)  $\int f d\mu_n \rightarrow \int f d\mu$  for every  $f \in C_\psi(\mathbb{R})$ .
- (iii)  $\int f d\mu_n \rightarrow \int f d\mu$  for every continuous  $f$  with compact support and for  $f = \psi$ .
- (iv)  $\mu_n \rightarrow \mu$  weakly and  $\int \psi d\mu_n \rightarrow \int \psi d\mu$ .

The following lemma gives a transparent characterization of the  $\psi$ -weakly compact subsets of  $\mathcal{M}_1^\psi$ . Recall that a set  $\mathcal{N} \subset \mathcal{M}_1^\psi$  is called uniformly  $\psi$ -integrating if it satisfies (2.13).

**Lemma A.4.** *A set  $\mathcal{N} \subset \mathcal{M}_1^\psi$  is relatively compact for the  $\psi$ -weak topology if and only if there exists a measurable function  $\phi : \mathbb{R} \rightarrow [0, \infty)$  such that  $\phi(x)/\psi(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$  and such that*

$$(A.2) \quad \sup_{\mu \in \mathcal{N}} \int \phi d\mu < \infty.$$

*In this case,  $\mathcal{N}$  is uniformly  $\psi$ -integrating.*



*Proof.* The first statement is an immediate consequence of Corollary A.47 in [18]. To prove the latter statement, we assume without loss of generality that  $\phi > 0$  and that  $\psi(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$ . Fix  $\varepsilon > 0$ , and denote by  $K$  the left-hand side of (A.2). Choosing  $M_1 > 0$  so large so that  $\psi(x)/\phi(x) \leq \varepsilon/K$  when  $|x| \geq M_1$ , and choosing  $M_0 > 0$  so large so that  $\psi(x) \geq M_0$  implies  $|x| \geq M_1$ , we obtain

$$\begin{aligned} \sup_{\mu \in \mathcal{N}} \int \psi(x) \mathbb{1}_{\psi(x) \geq M} \mu(dx) &= \sup_{\mu \in \mathcal{N}} \int \phi(x) \frac{\psi(x)}{\phi(x)} \mathbb{1}_{\psi(x) \geq M} \mu(dx) \\ &\leq \frac{\varepsilon}{K} \sup_{\mu \in \mathcal{N}} \int \phi(x) \mu(dx) \\ &= \varepsilon \end{aligned}$$

for all  $M \geq M_0$ . That is, (2.13) holds.  $\square$

## References

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